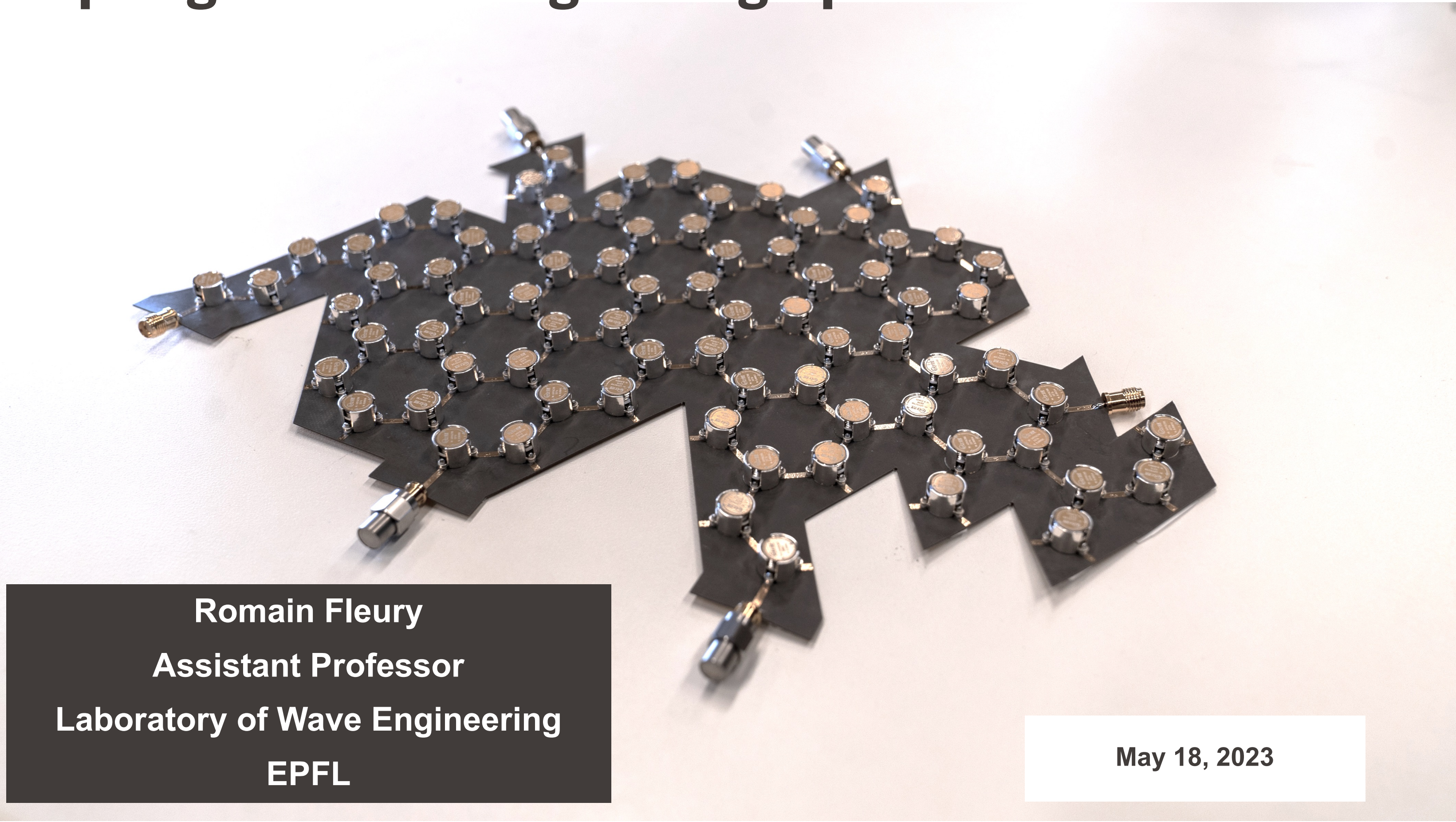
Topological scattering: from graphs to networks



Romain Fleury

Assistant Professor

Laboratory of Wave Engineering

EPFL

May 18, 2023

École
polytechnique
fédérale
de Lausanne

EPFL at glance

12’576
Students

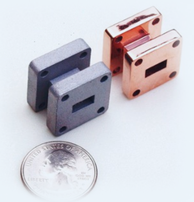
366
Professors

+6’400
Employees (incl. Phd)

+130
Nationalities

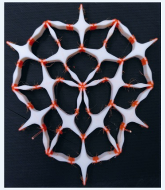
Research interests

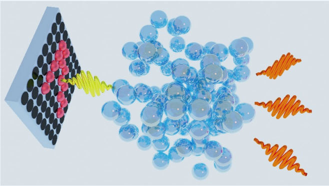
Subwavelength wave control



Miniaturized RF devices
Physical Review Applied (2021)
MinWave start-up (2021)

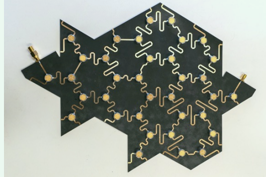
Nonlocal (elastic) metamaterials
Phys. Rev. Lett. (2023)
arXiv 2209.02618





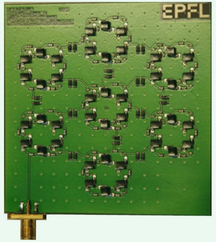
Sub-λ acoustic imaging
Physical Review X (2019)

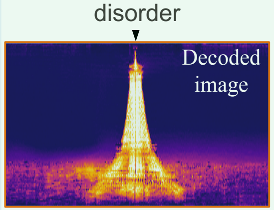
Topologically robust wave devices



One-way RF transport
Nature (2022)
Science Advances (2023)

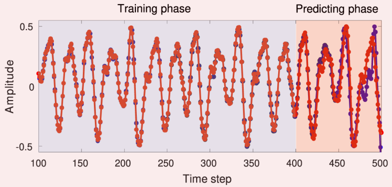
Protected resonances
Phys. Rev. Lett. (2019)
Phys. Rev. Lett. (2019)



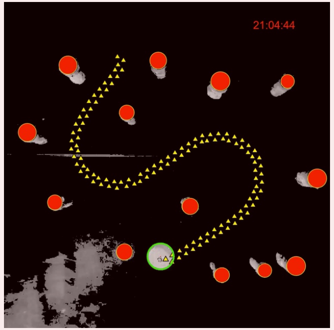


Robust signal processing
Nature Communications (2019)
Advanced Materials (2023)

Active wave systems



Wave-based Neuromorphic computing,
Nature Communications (2022)
arXiv 2304.11042 (2023)



Active sound control
Nature Physics (2018)
Phys. Rev. Applied (2019)

Team and collaborators

Ph.D. students: Zhe Zhang, Qialu Chen, Aleksi Bossart, Rongrong Xiang, Mathieu Padlewski, Ali Momeni, Tinggui Chen.
Postdocs: Matthieu Malléjac, Janez Rus, Maliheh Khatibi, Benjamin Apffel

Zhe Zhang, Ph.D.
2019-now

Pierre Delplace, CNRS
ENS de Lyon (France)

erc FNSNF HASLERSTIFTUNG

Introduction to topology

Topological scattering

From graphs to topological networks

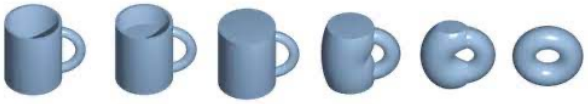
Conclusion and perspectives

What is topology?

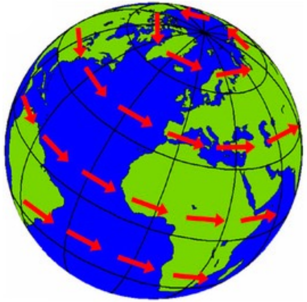
Topology = mathematics of continuous transformations

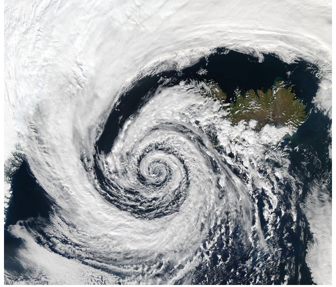
study of topological equivalences

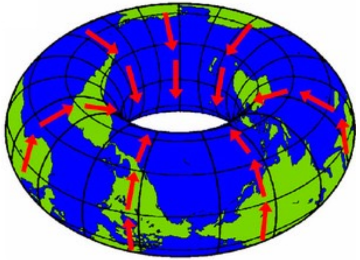
described by topological invariants

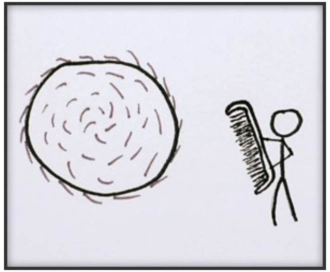


Topology garantees some properties



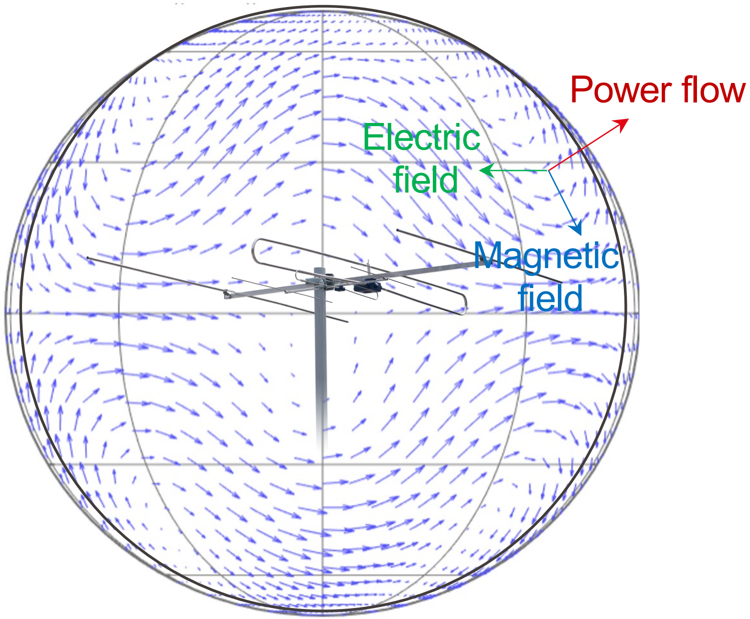




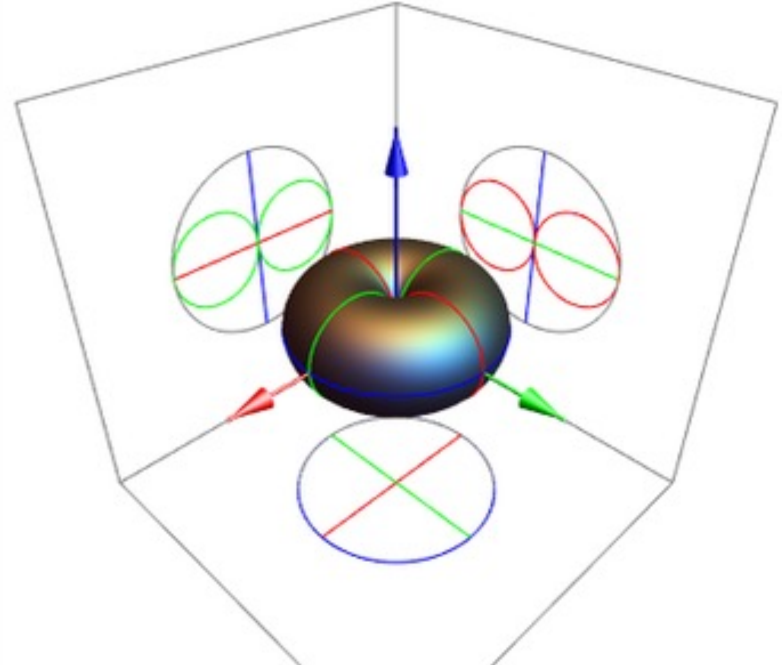


$$2(1-g)=\sum v$$

Example in electromagnetism

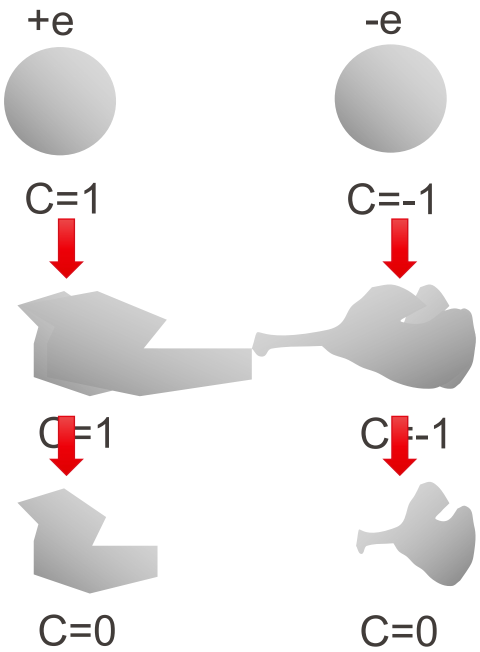


Magnetic far field
Tangential vector bundle



Field vortices = dead zones
cannot be avoided !
=no monopolar radiation

Example in electrostatics



+e

-e

C=1

C=-1

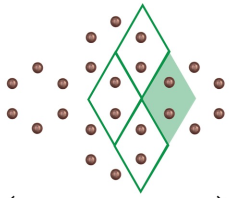
C=1

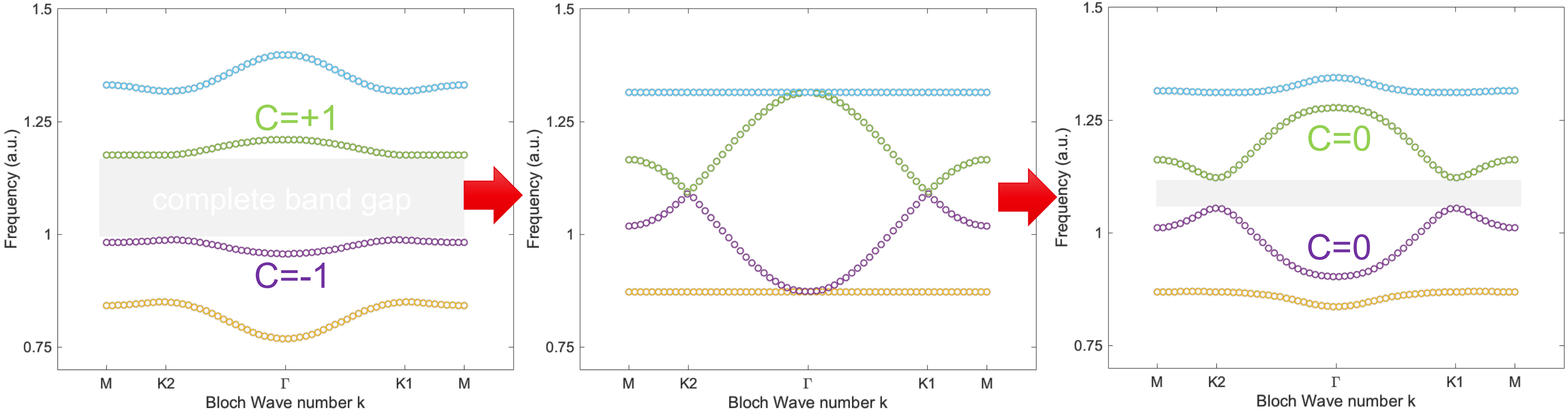
C=-1

C=0

C=0

Periodic structures and topological charges

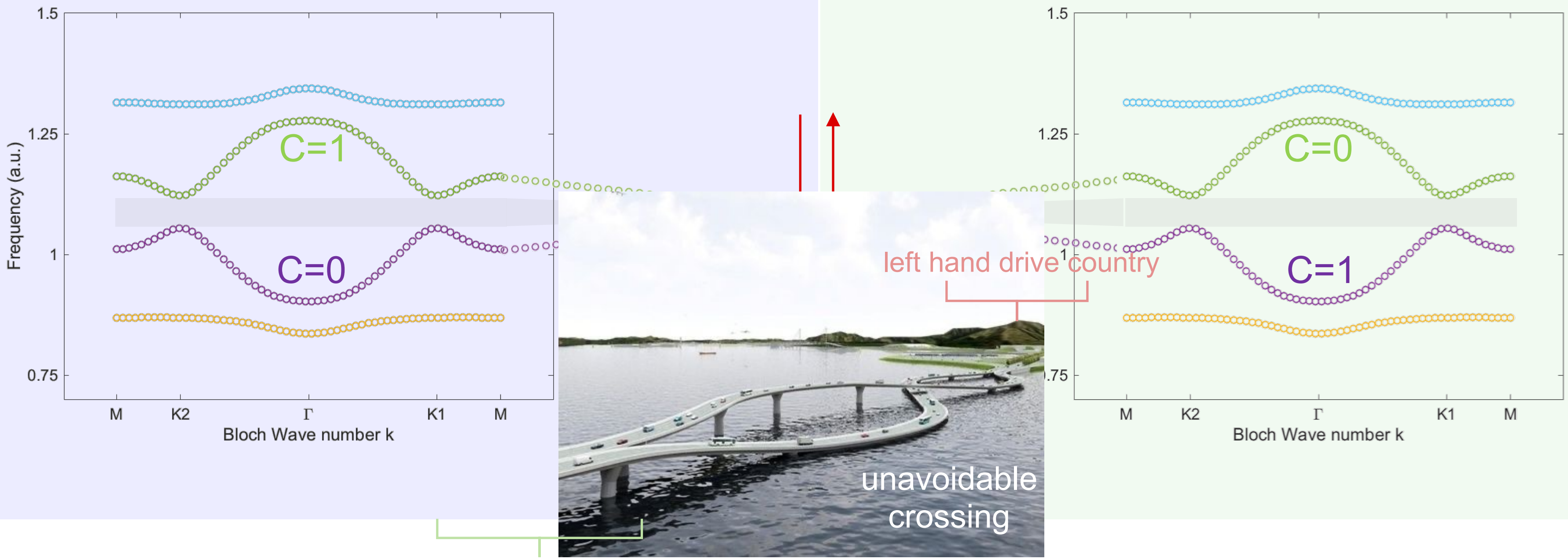




Topological edge modes

Topology #1

Topology #2

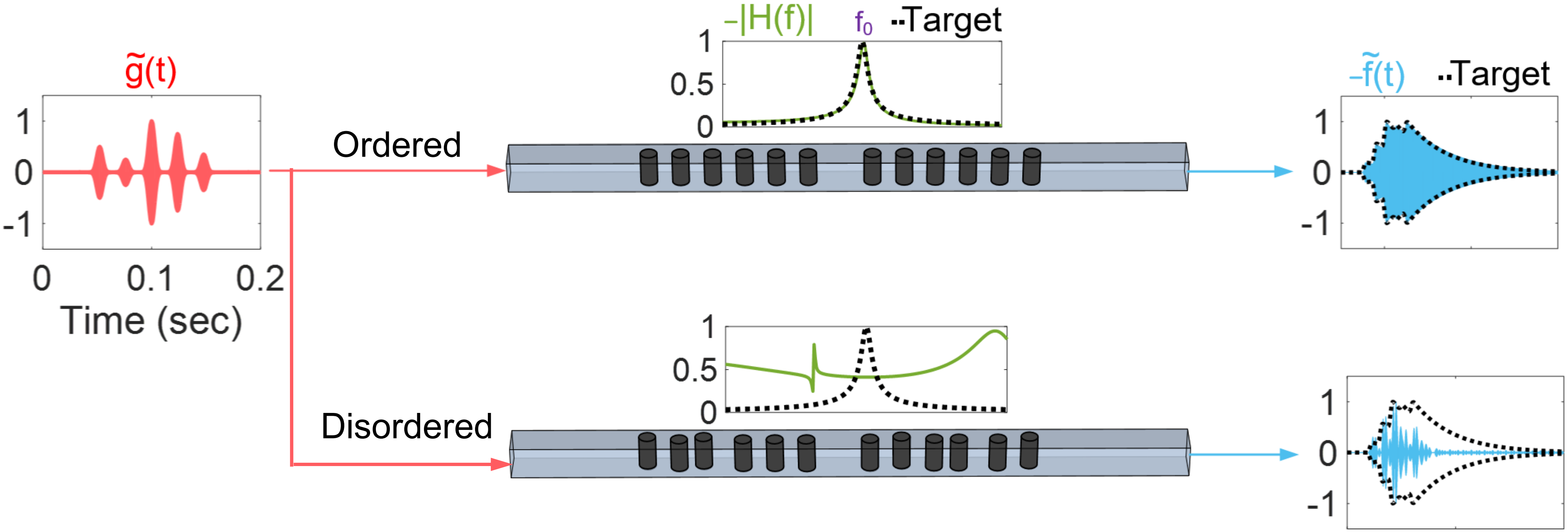


left hand drive country

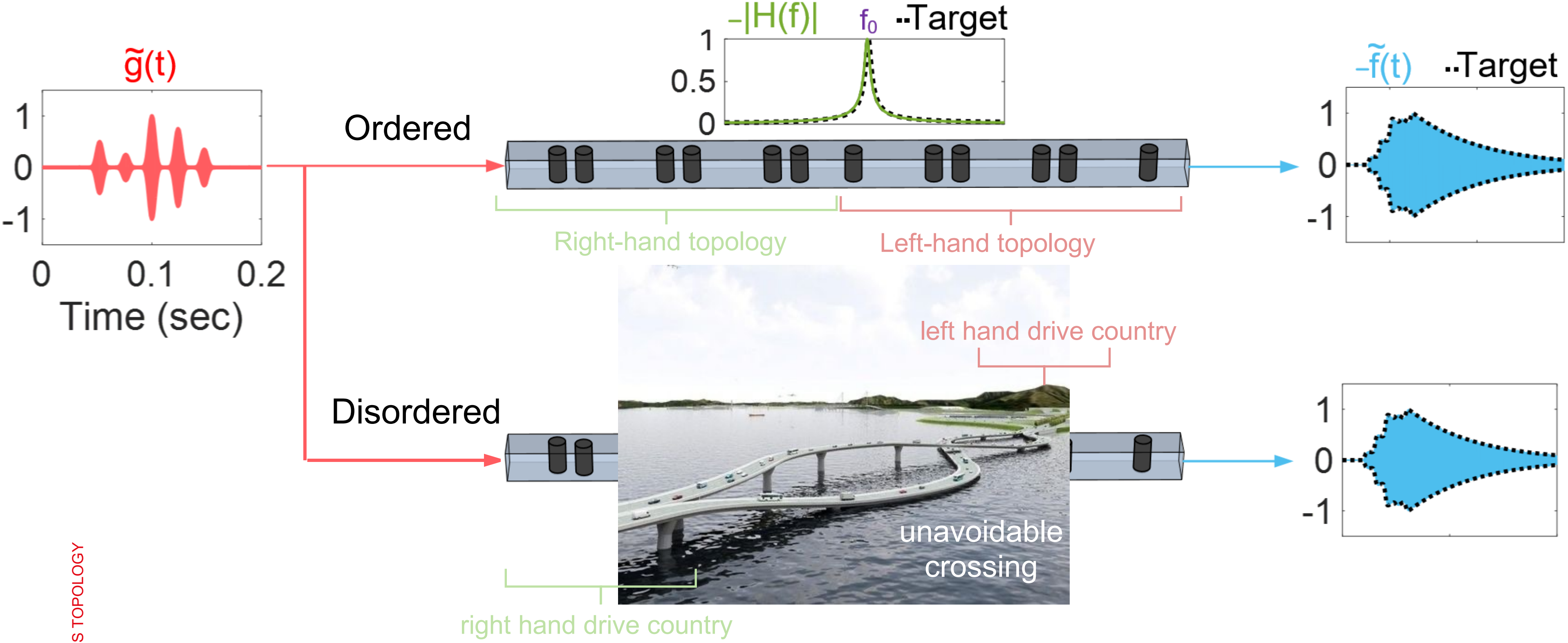
unavoidable crossing

right hand drive country

Application to robust scattering ?

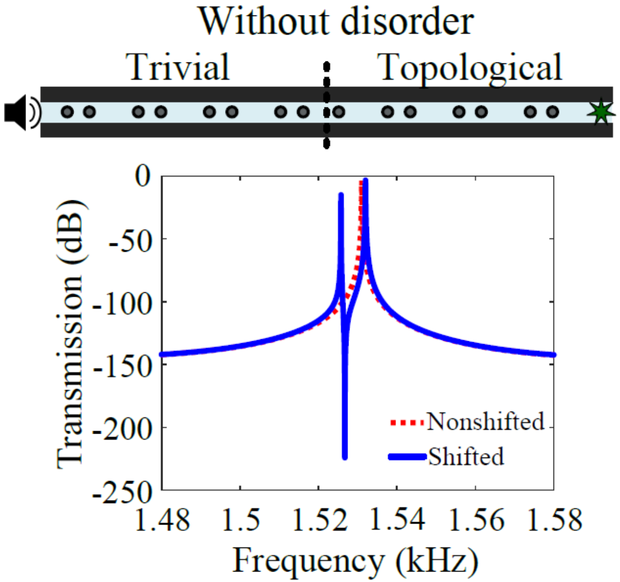


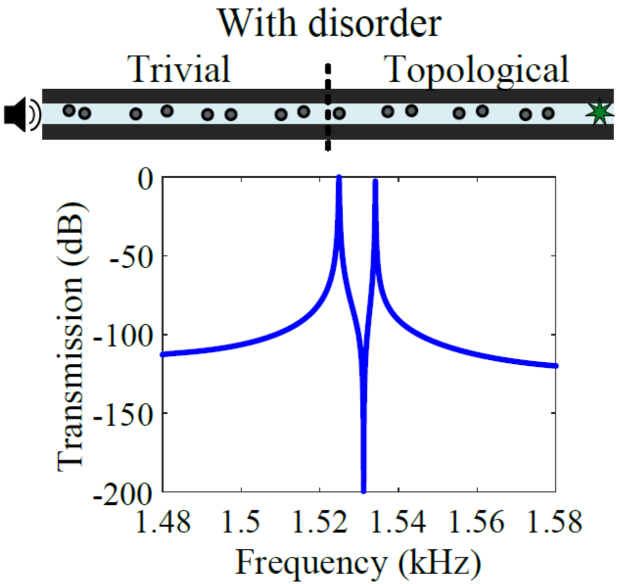
Topologically robust scattering poles

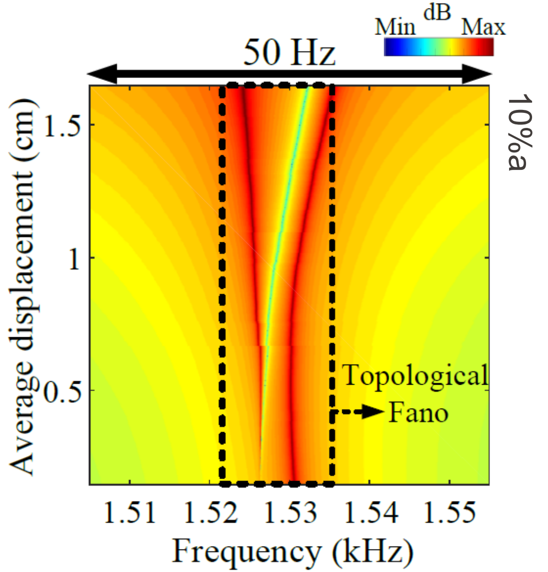


Zangeneh, Fleury, Nature Communications 10, 2058 (2019); Phys. Rev. Lett. 122, 014301 (2019); Advanced Materials 32, 2001034 (2020); Optics Letters 45, 5966 (2020); Phys. Rev. B 101, 024101 (2019)

Topological Fano resonance

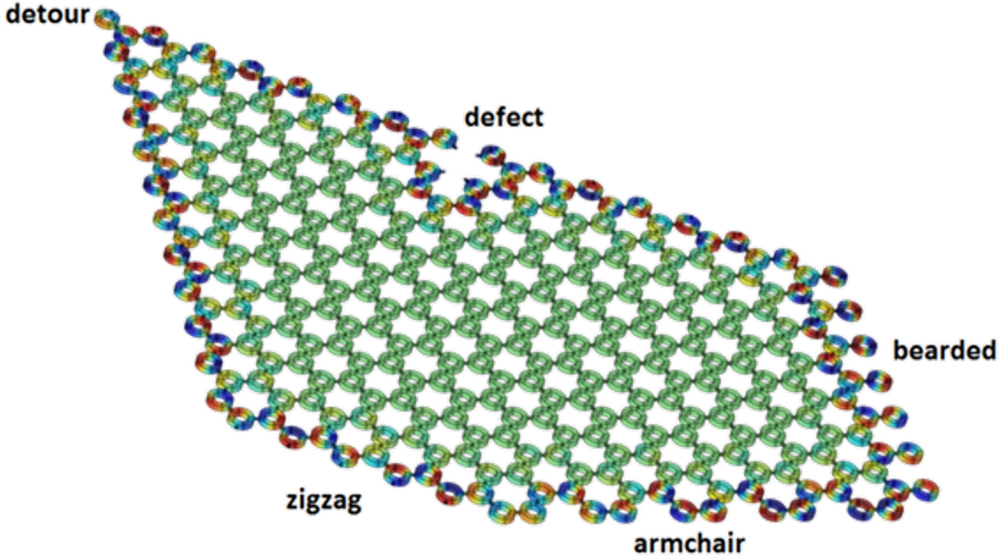


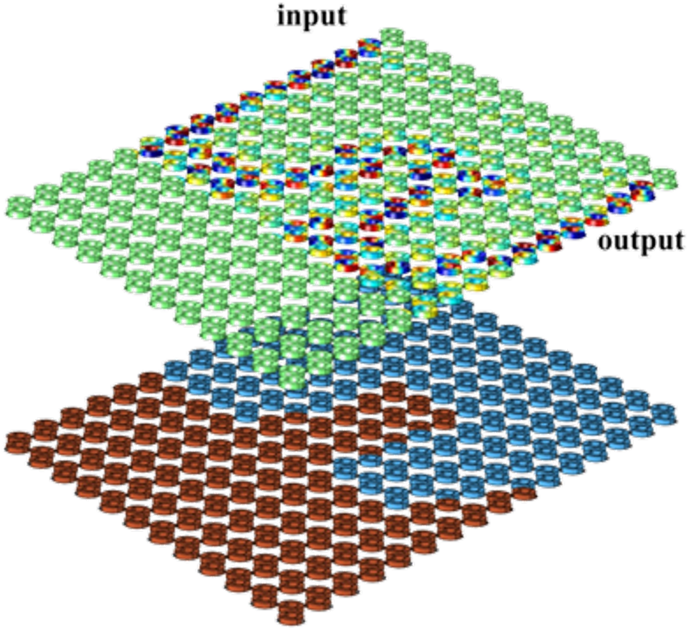




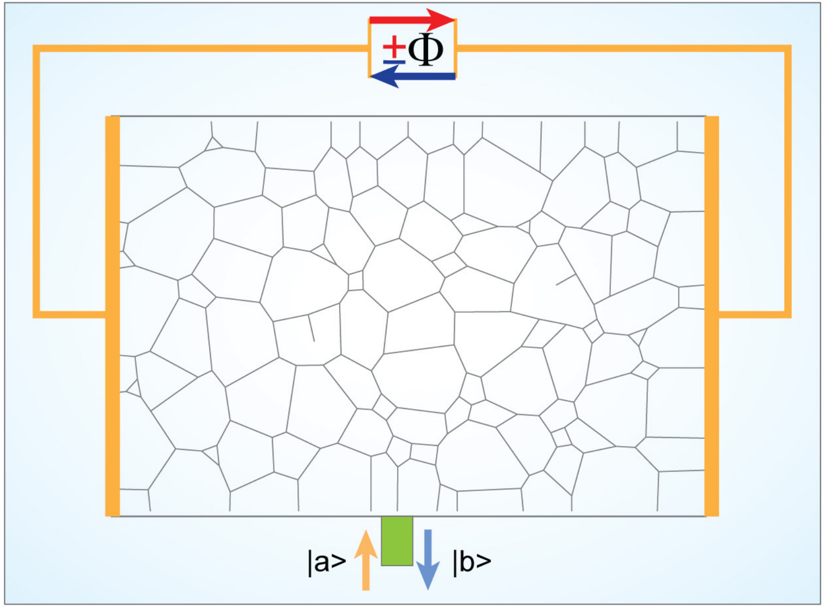
Zangeneh, Fleury, Phys. Rev. Lett. 122, 014301 (2019)

2D topological insulators





Nat. Comm. **6**, 8260 (2015)



Introduction to topology

Topological scattering

From graphs to topological networks

Conclusion

How to define topological charges ?
The notion of Homotopy invariants

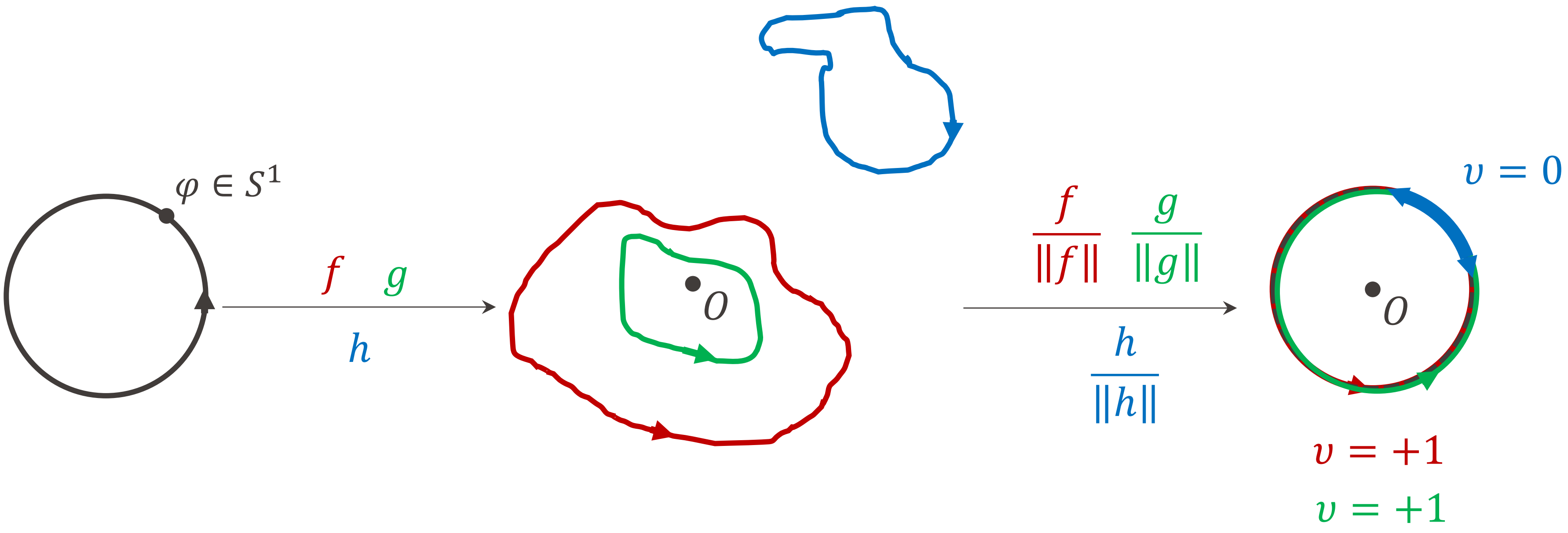
Continuous maps:

$$S^1 \xrightarrow{f \;\simeq\; g \;\not\simeq\; h} \mathbb{R}^2 \setminus \{O\}$$

$$\varphi \longrightarrow f(\varphi)$$

Homotopy classes characterized by the winding number of the map = homotopy invariant

Winding number



$\varphi \in S^1$

$\xrightarrow[h]{f \quad g}$

O

$\xrightarrow[\frac{h}{\|h\|}]{\frac{f}{\|f\|} \quad \frac{g}{\|g\|}}$

O

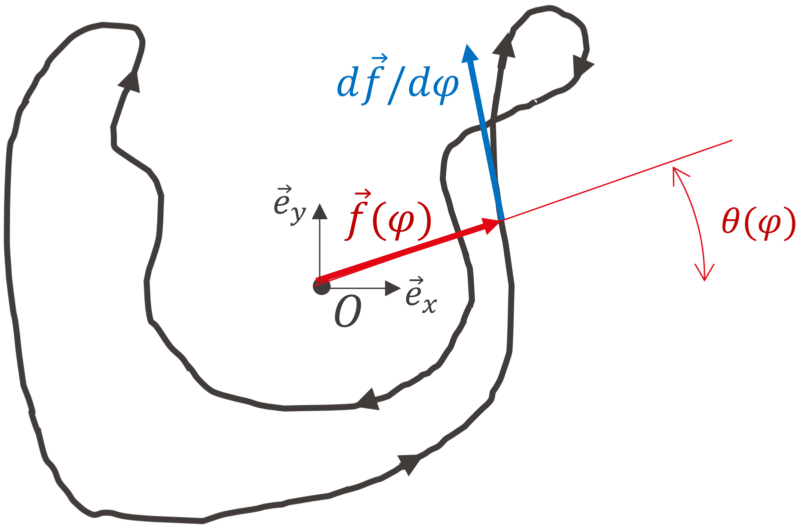
$\upsilon = 0$

$\upsilon = +1$

$\upsilon = +1$

$$S^1 \bullet\!\!\longrightarrow \mathbb{R}^2 \setminus \{O\} \bullet\!\!\longrightarrow S^1$$

The winding number of the map



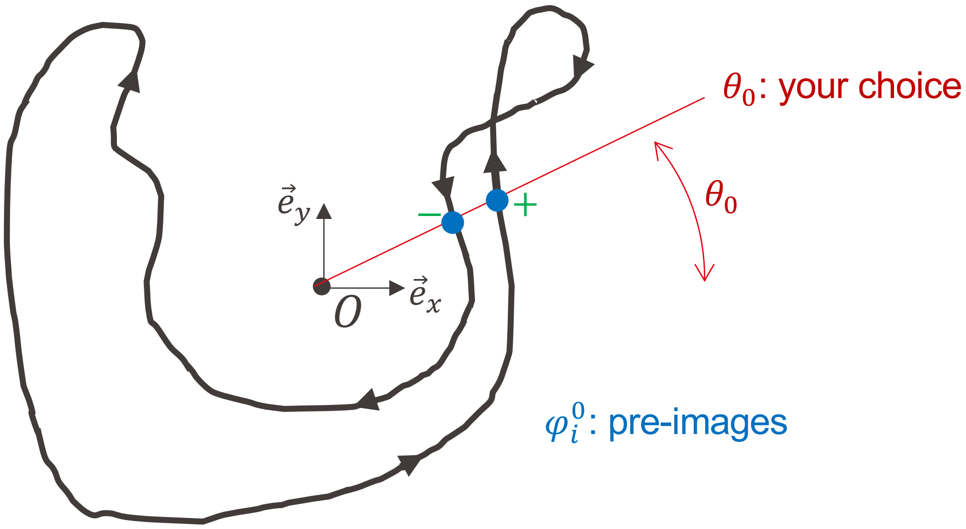
$\left(\vec{f}\times\frac{d\vec{f}}{d\varphi}\right).\vec{e}_z$ Orbital moment, do I go right or left ?

$$\boxed{\upsilon(f)=\frac{1}{2\pi}\oint_{S^1}d\varphi\,\frac{\left(\vec{f}\times\frac{d\vec{f}}{d\varphi}\right).\vec{e}_z}{\|\vec{f}\|^2}}$$ Winding number

Proof: $\vec{f}=\|\vec{f}\|\begin{pmatrix}\cos\theta(\varphi)\\ \sin\theta(\varphi)\end{pmatrix}$ $\Longrightarrow \upsilon(f)=\frac{1}{2\pi}\oint_0^{2\pi}d\theta\in\mathbb{Z}$

($\upsilon=0$ here)

More practical definition

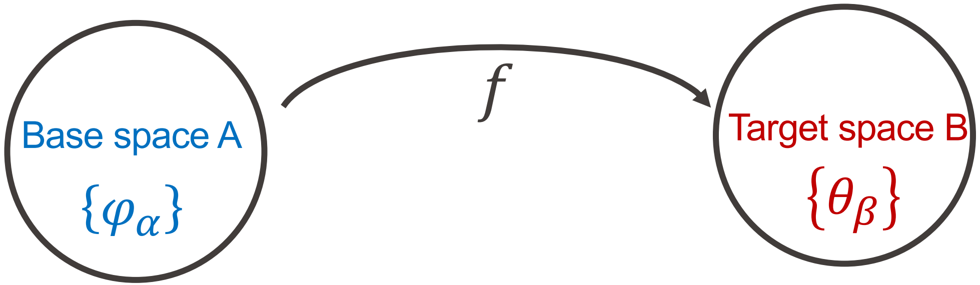


Winding number:

$$\upsilon(f) = \sum_{\substack{\varphi_i^0 \text{ such that} \\ \mathrm{Arg}\left(f\left(\varphi_i^0\right)\right)=\theta_0}} \operatorname{sgn}\left(\vec{f}\times\frac{d\vec{f}}{d\varphi}\right).\vec{e}_z$$

$$\upsilon = -1 + 1 = 0$$

Generalization : the degree of a map



Base space A

$\{\varphi_\alpha\}$

f

Target space B

$\{\theta_\beta\}$

A,B orientable

A,B: S^1, S^2, T^2, etc

$\dim A = \dim B$

$$\boxed{\deg f \;=\sum_{\varphi_i^0} \operatorname{sgn}\det\left(\frac{\partial\theta_\beta}{\partial\varphi_\alpha}\right)_{f(\varphi_i^0)=\theta_i^0}}$$

Ex: The winding number $\nu(f) = \deg f$ where f is a map from S^1 to S^1

$$\prod_1 S^1 = \mathbb{Z}$$

Homotopy group from S^1 to S^1

Homotopy groups from S^n to S^m

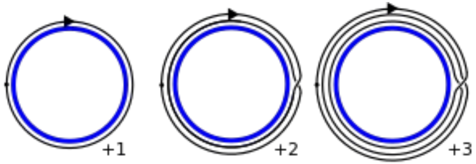
$\prod_n S^m = \ \ldots$

0 when $n < m$ (trivial all maps can be deformed into each other)

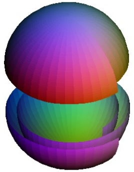
$\prod_1 S^2$

$\mathbb{Z}$ when $n = m$ (the case we've seen)

$\prod_1 S^1$



$\prod_2 S^2$



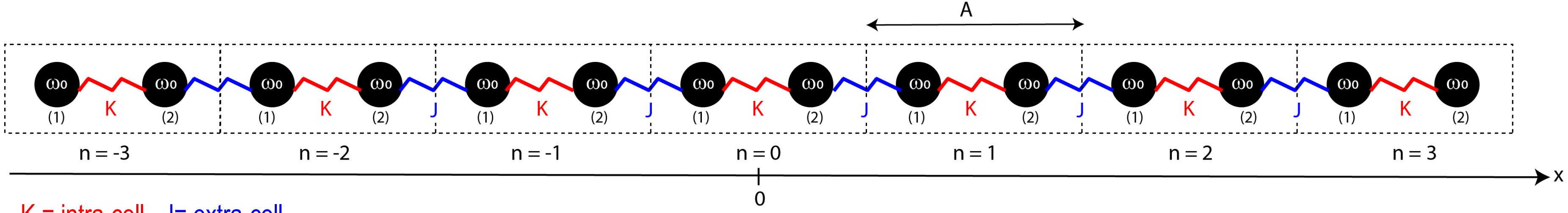
$n > m$ it depends, open question in general

Ex: $\prod_3 S^2 = \mathbb{Z}$, Hopf insulators $\prod_{10} S^3 = \mathbb{Z}_{15}$ ...

Source: Wikipedia, "homotopy groups of spheres"

Application to 1D topological insulators

Infinite chain of identical resonators with 2 couplings **(K, J)** and of period **A** (**2 resonators in the unit cell**)



K = intra-cell   J= extra-cell

Coupled harmonic oscillators equations + Bloch theorem → Tight-binding **Hamiltonian** describing the chain

K_{tot}

Resonance 1

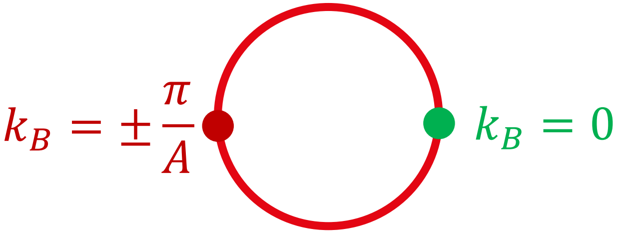
$$H_{TB}(k_B) = \begin{bmatrix} \omega_0 & K + Je^{-jk_BA} \\ K + Je^{jk_BA} & \omega_0 \end{bmatrix}$$

K_{tot}^*

Resonance 2

Before defining topology - Things to keep in mind

1)The Bloch wave number lives on a circle : $k_B \in \left[-\frac{\pi}{A}, \frac{\pi}{A}\right]$

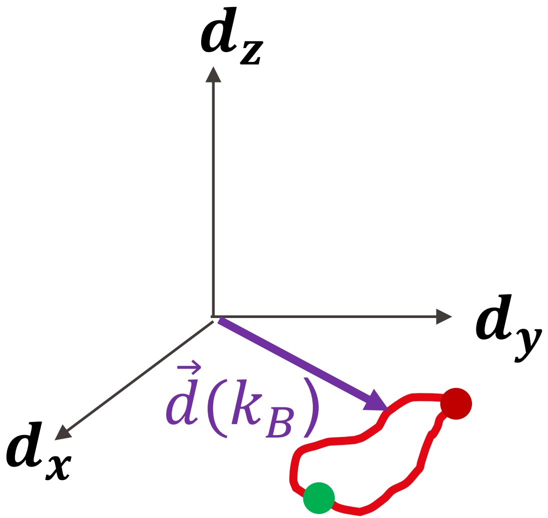


2) The Hamiltonian lives in the space of 2 by 2 Hermitian matrices

$$H_{TB}(k_B) = d_0\boldsymbol{\sigma_0} + d_x(k_B)\boldsymbol{\sigma_x} + d_y(k_B)\boldsymbol{\sigma_y} + d_z(k_B)\boldsymbol{\sigma_z}$$

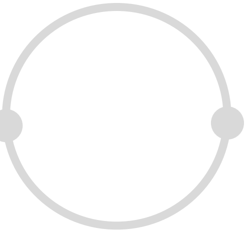
3) The eigenvalues of the Hamiltonian give the two frequency bands

$$\omega_\pm = d_0 \pm \left|\vec{d}(k_B)\right|$$



Before defining topology - Things to keep in mind

1)The Bloch wave number lives on a circle : $k_B \in \left[-\frac{\pi}{A}, \frac{\pi}{A}\right]$



$k_B = \pm\frac{\pi}{A}$ $k_B = 0$

2) The Hamiltonian lives in the space of 2 by 2 Hermitian matrices (closed path parametrized by k_B)

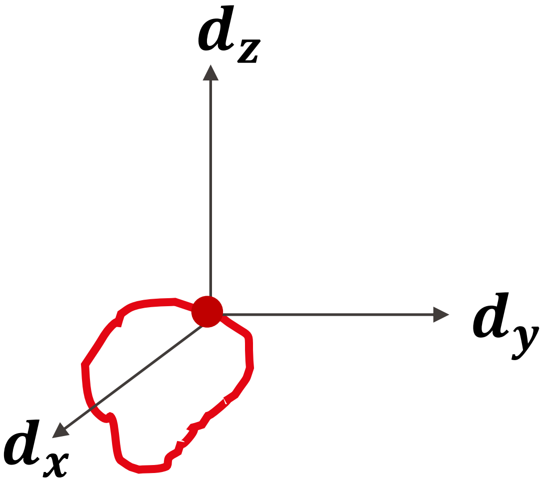
$$H_{TB}(k_B) = d_0\boldsymbol{\sigma_0} + d_x(k_B)\boldsymbol{\sigma_x} + d_y(k_B)\boldsymbol{\sigma_y} + d_z(k_B)\boldsymbol{\sigma_z}$$

3) The eigenvalues of the Hamiltonian give the two frequency bands

$$\omega_\pm = d_0 \pm \left|\vec{d}(k_B)\right|$$

4) The band gap closes when $\left|\vec{d}(k_{B,close})\right| = 0$

Restricting to insulators excludes the origin



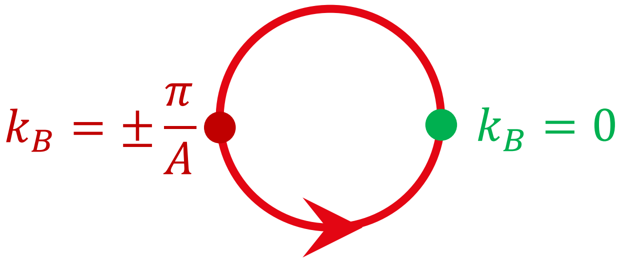
Defining topology under chiral symmetry

$$H_{TB}(k_B) = d_0\boldsymbol{\sigma_0} + d_x(k_B)\boldsymbol{\sigma_x} + d_y(k_B)\boldsymbol{\sigma_y} + \cancel{d_z(k_B)}\boldsymbol{\sigma_z}$$

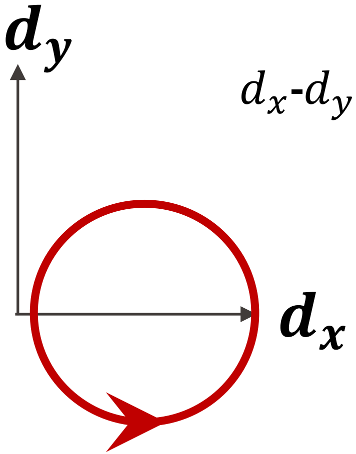
identical resonators = chiral symmetry

$$d_x(k_B) = \Re(K_{tot}) = K + J\cos k_B A$$
$$d_y(k_B) = \Im(K_{tot}) = J\sin k_B A$$

Brillouin zone



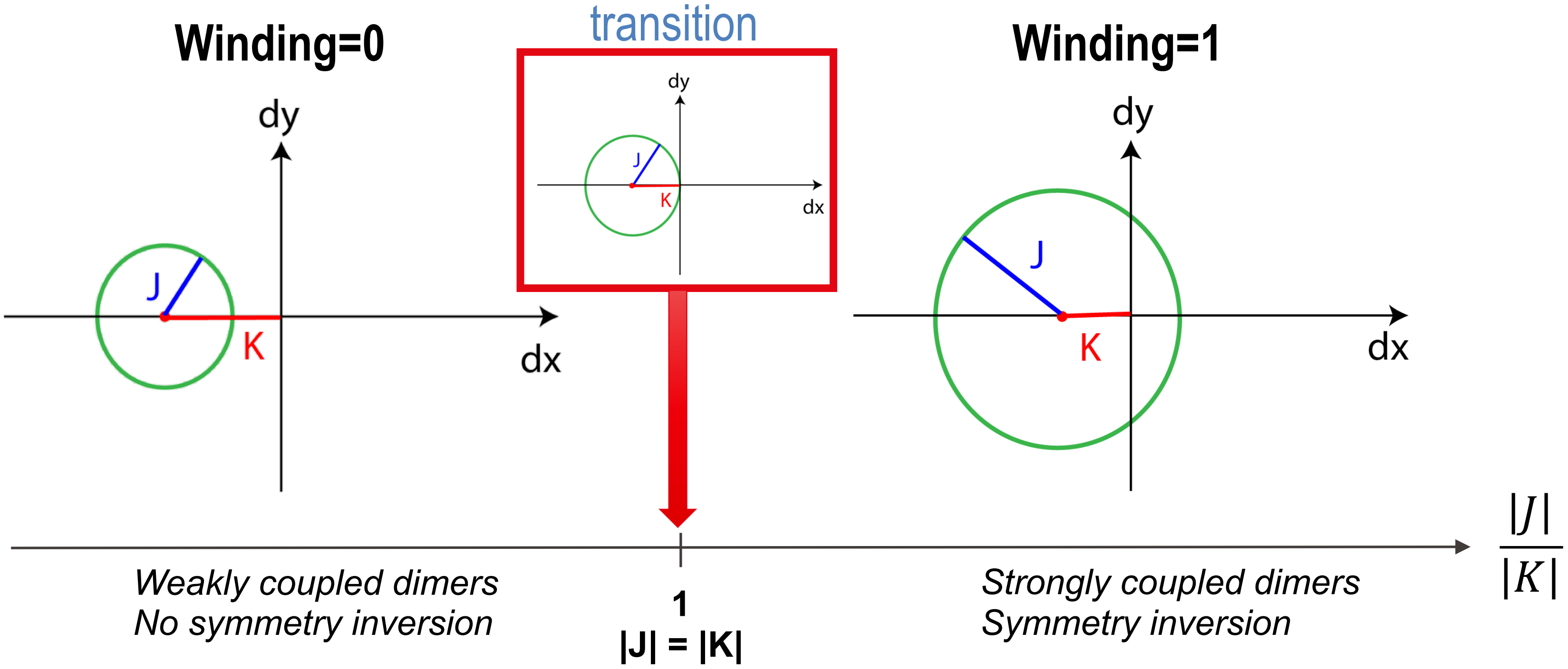
d_x-d_y plane



$$\prod_1 S^1 = \mathbb{Z}$$

Homotopy group from S^1 to S^1

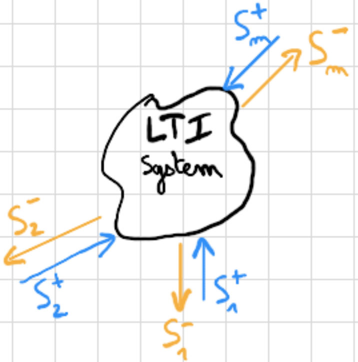
The winding number as a topological invariant



Weakly coupled dimers
No symmetry inversion

Strongly coupled dimers
Symmetry inversion

Scattering matrices: textbook definition



LTI : linear time-invariant,

$S_i^+ \in \mathbb{C}$: incident wave amplitude at port $i \in [\![1, m]\!]$

$S_i^- \in \mathbb{C}$: outgoing ___ ____ ______

Convention : $S_i^\pm$ is normalized such that $|S_i^\pm|^2 = {S_i^\pm}^{+} S_i^\pm$ is the incident/outgoing power at port i.

Linearity $\Rightarrow$ $\exists\, S\, /$ $\boxed{\vec{S}_- = S \vec{S}_+}$ m × m scattering matrix

$\vec{S}_\pm = [S_1^\pm, S_2^\pm, \dots, S_m^\pm]^T$ m × 1 column vector

Goal: define topology directly from S

Unitarity of S

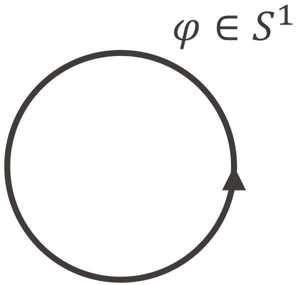
S conserves power flux (ex: lossless system) $\Leftrightarrow$ S is unitary: $(S^\dagger S) = \mathbb{I}_m,\ S \in U(m)$

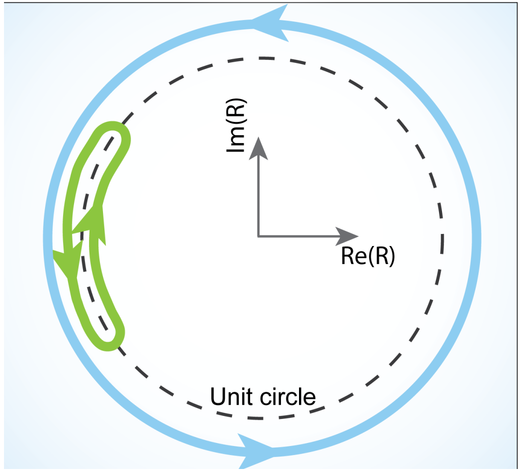
Proof: For all input S_+ , $S_+^\dagger S_+ \ = S_-^\dagger S_-$ $\Leftrightarrow S_+^\dagger S_+ \ = S_+^\dagger (S^\dagger S) S_+$ $\Leftrightarrow S_+^\dagger (S^\dagger S - \mathbb{I}_m) S_+ \ = 0$

One port example: $\mathrm{S} = R = e^{j\theta} \in U(1)$

Two port example: $$\mathrm{S} = \begin{pmatrix} j\cos\theta & \sin\theta \\ \sin\theta & j\cos\theta \end{pmatrix} \in U(2)$$

Homotopy of unitary matrices





Winding of reflection phase

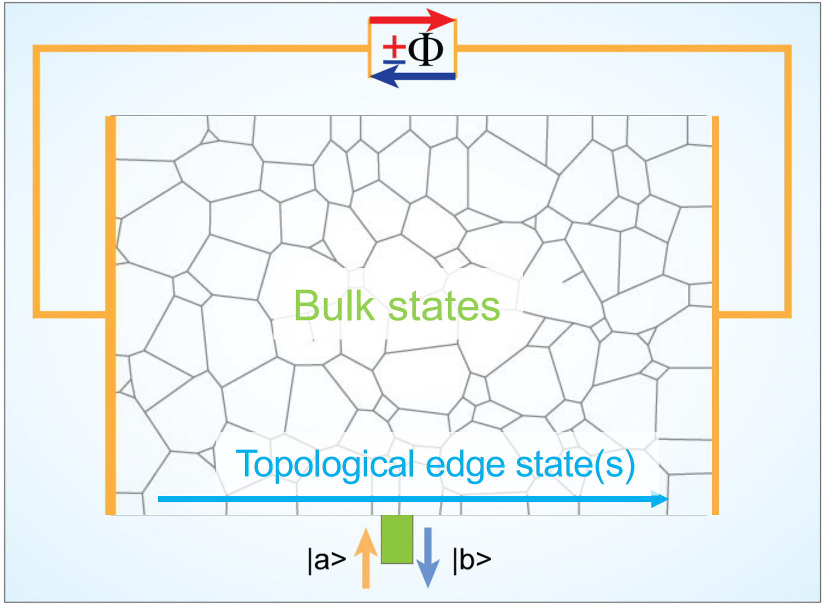
$$S^1 \xrightarrow{\quad S \quad} U(1)$$

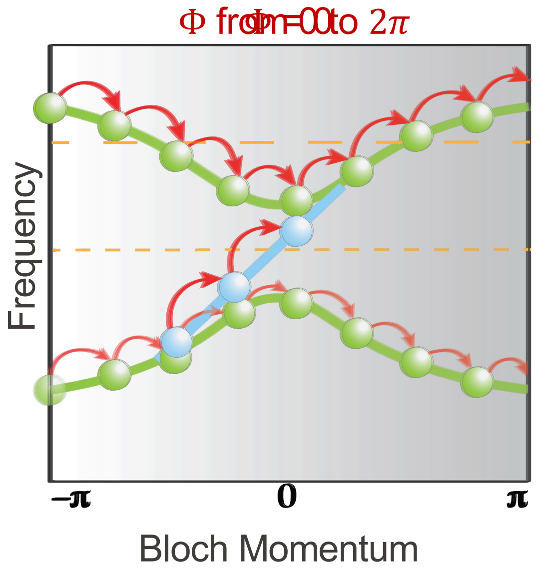
$$\varphi \longrightarrow S(\varphi) = R = e^{j\theta(\varphi)}$$

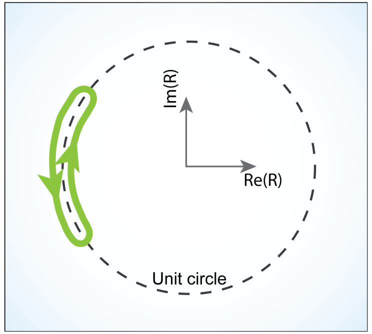
Homotopy group and invariant:

$$\Pi_1(U[m]) = \mathbb{Z} \quad \Rightarrow \upsilon(S) = \frac{1}{2\pi}\oint_0^{2\pi} d\theta(\varphi) \quad \Rightarrow \upsilon(S) = \frac{1}{2\pi j}\oint_0^{2\pi} d\varphi \frac{d \ln \det S}{d\varphi} = \frac{1}{2\pi j}\oint_0^{2\pi} d\varphi \, Tr(S^\dagger \frac{dS}{d\varphi})$$

Scattering topological invariants in the lab







We can define and measure topological invariants from probe scattering.

- *Topological pumps and scattering: See works by Laughlin, Akhmerov, Brower, Nascimbene, Chong, Hafezi, and many others.*

Topological scattering transitions

Real excitation frequency

Coupled mode theory:
(Mahaux-Weidenmüller formula)

$$S(\omega) = C + D[j(\omega\mathbb{I} - H) + \Gamma]^{-1}K^T$$

Scattering matrix

Direct path

Closed system Hamiltonian

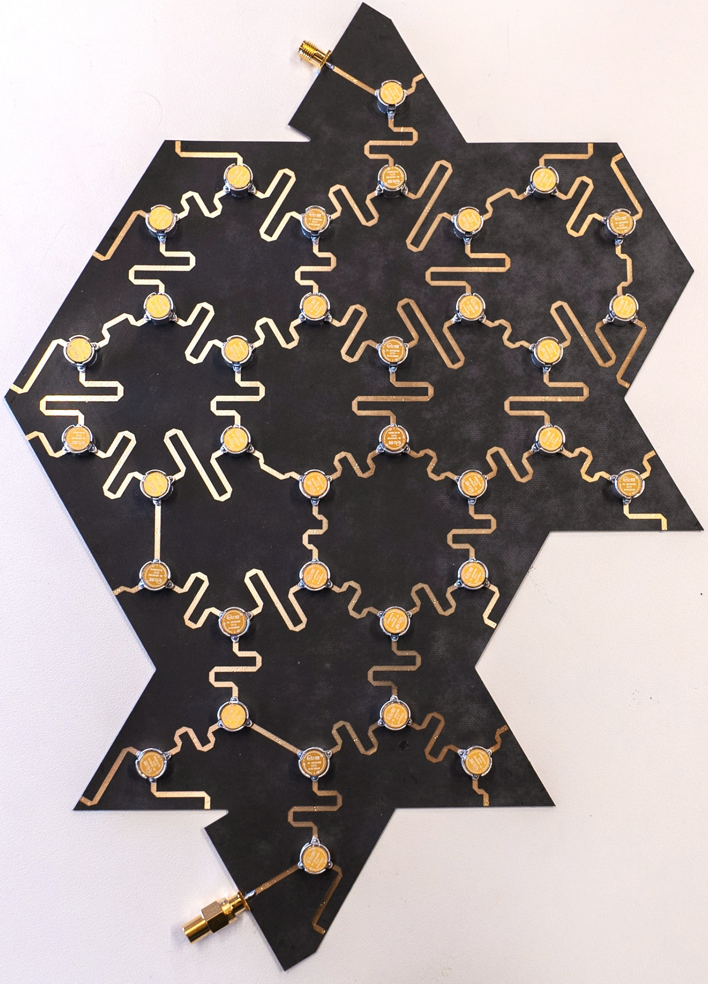
Decay matrix

K, D, Coupling matrices

Topological transition
=
Singularity

$\Longrightarrow j(\omega\mathbb{I} - H) + \Gamma$ non invertible $\Leftrightarrow \exists\psi_\infty$ such that $(H + j\Gamma)\,\psi_\infty = \omega\,\psi_\infty$

Topological transition = Bound state in continuum !



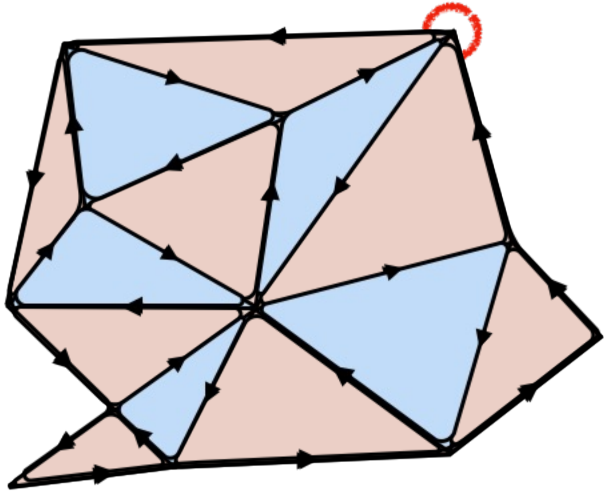
Introduction to topology

Topological scattering

From graphs to topological networks

Conclusion

Eulerian graphs

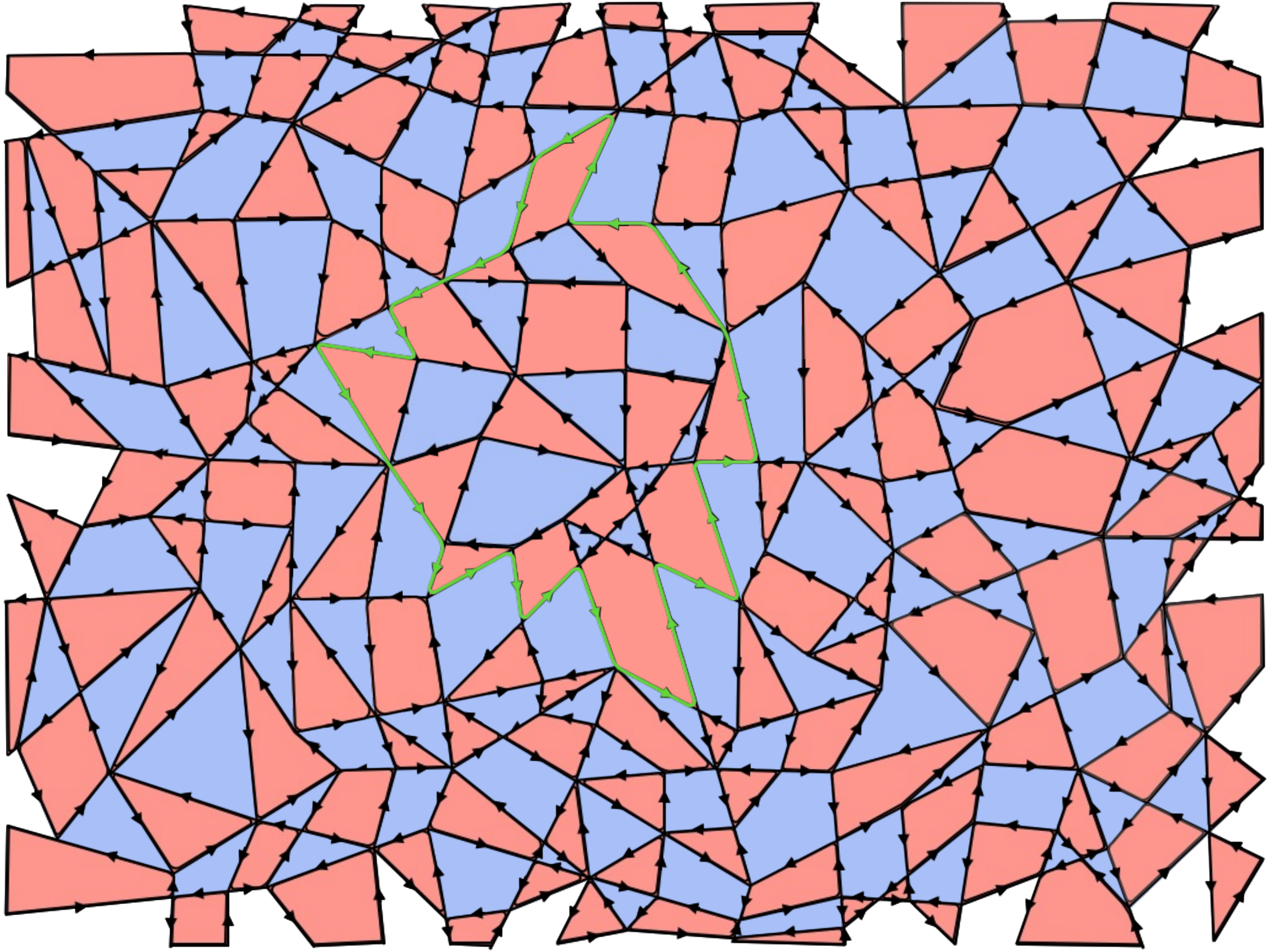


Non-Minimal Veblen decomposition = topological phase

Minimal Veblen decomposition = trivial phase

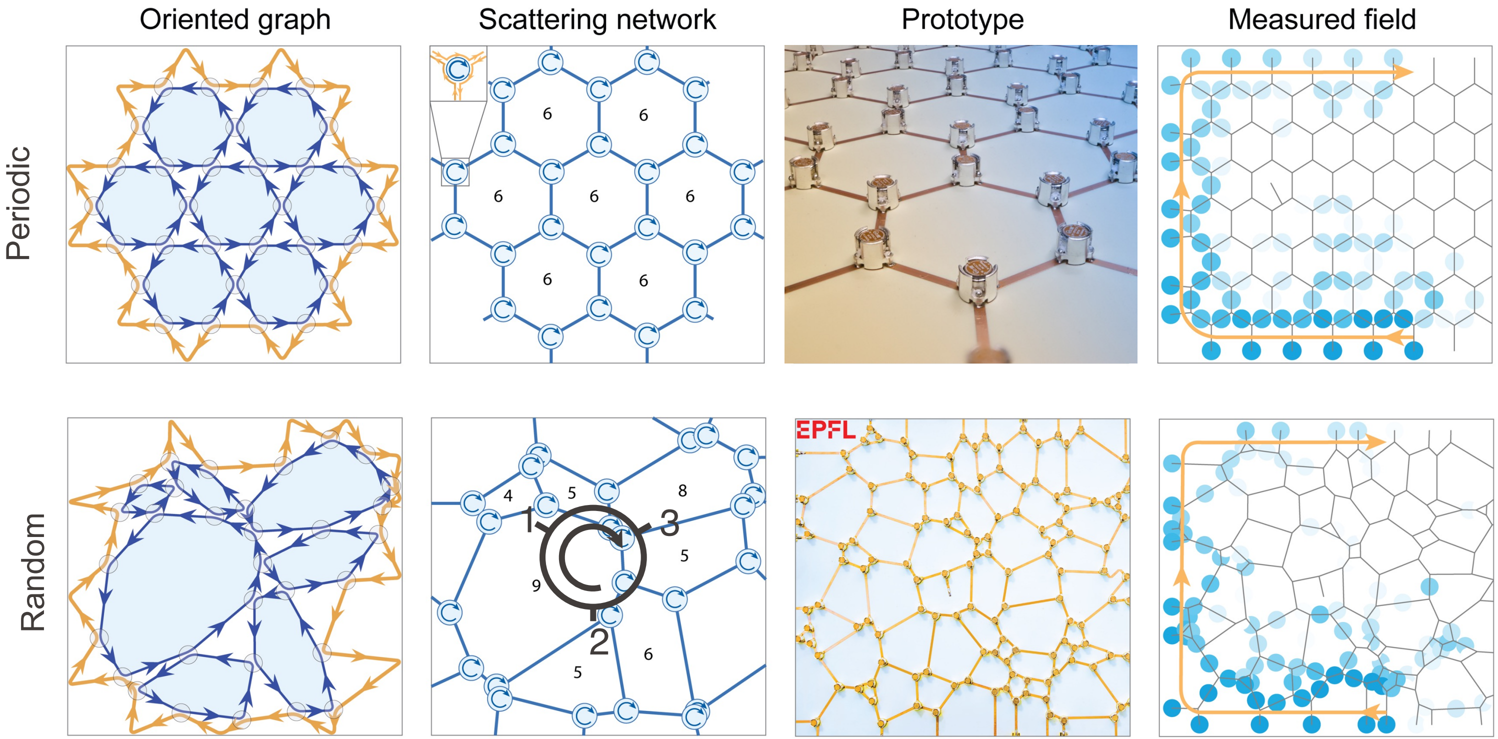
The seven bridges of Königsberg, Wikipedia.

Delplace, SciPost Phys. (2021), doi: 10.21468/SciPostPhys.8.5.081
Zhang, Delplace, Fleury, Science Advances (2023), DOI : 10.1126/sciadv.adg3186
Zhang, Delplace, Fleury, Nature (2021), DOI : 10.1038/s41586-021-03868-7



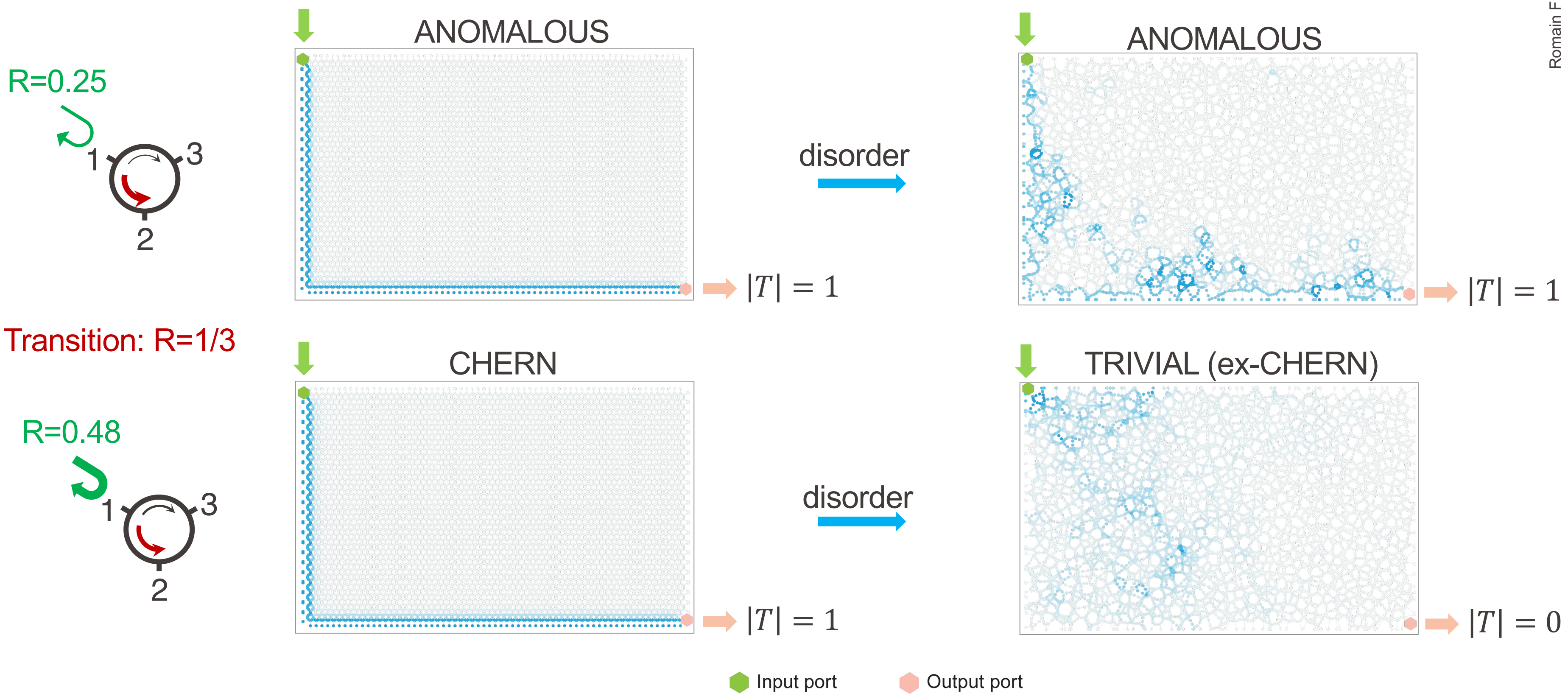
Delplace, doi: 10.21468/SciPostPhys.8.5.081

Practical Eulerian graphs : circulator networks



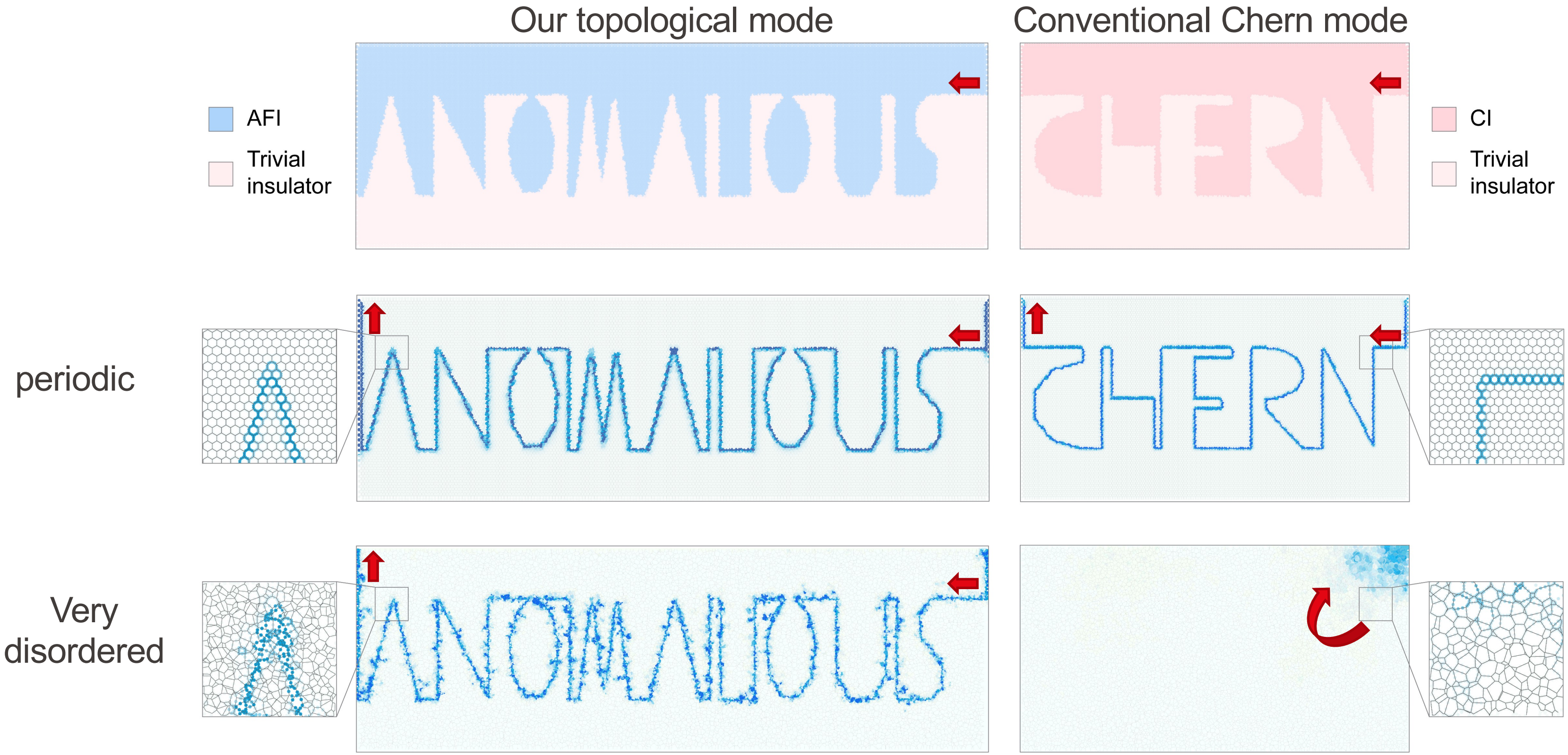
Zhang, Delplace, Fleury, Science Advances 9,eadg3186 (2023), Nature 598, 293 (2021)

The rich physics of non-reciprocal networks



We want to define and measure topological invariants from scattering measurements

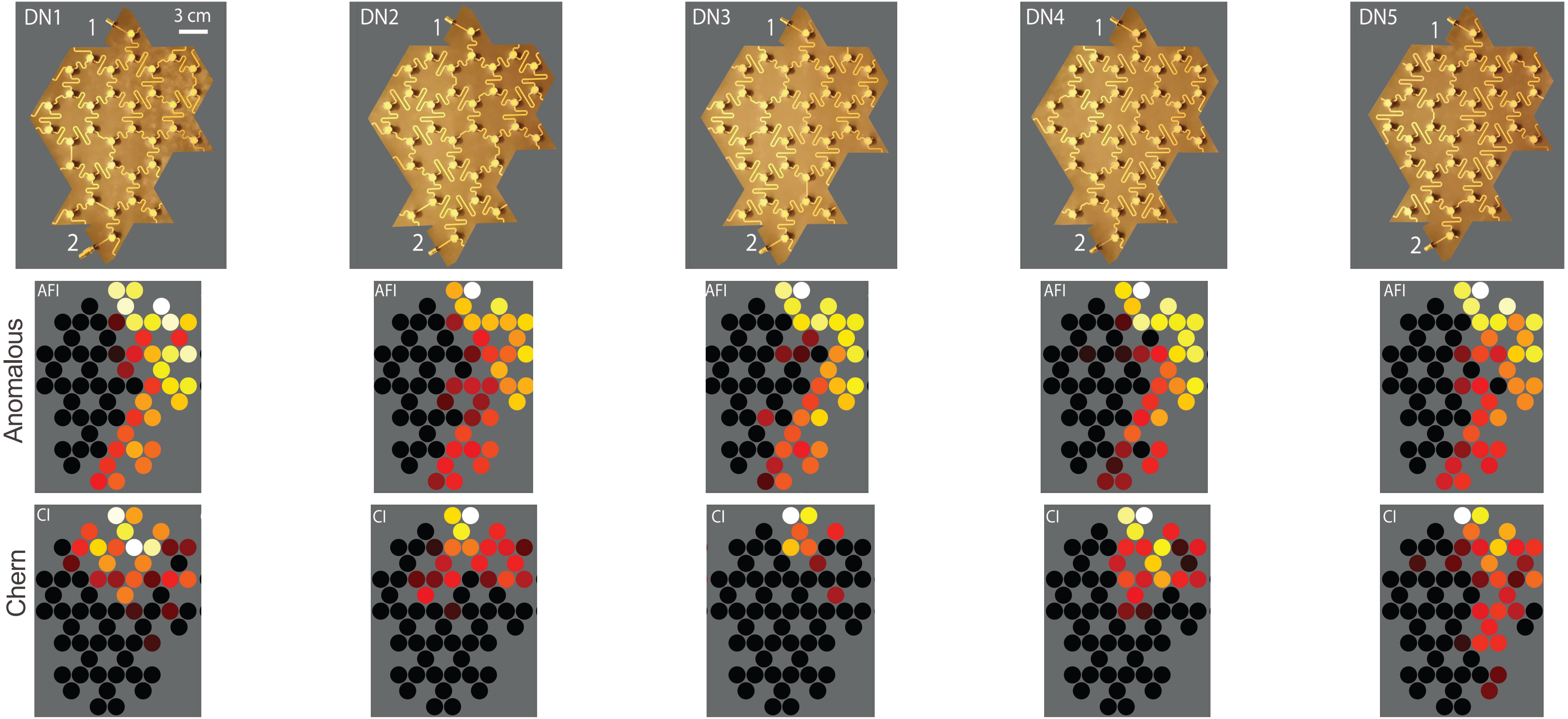
Anomalous immunity to large disorder



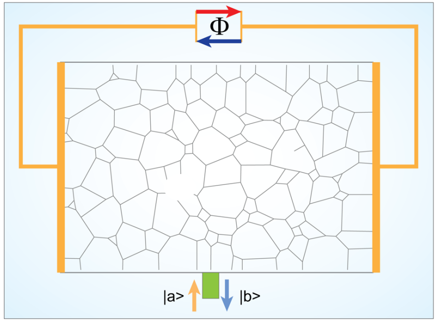
Zhang, Delplace, Fleury, Science Advances 9,eadg3186 (2023), Nature 598, 293 (2021)

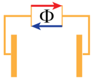
Other disorder types

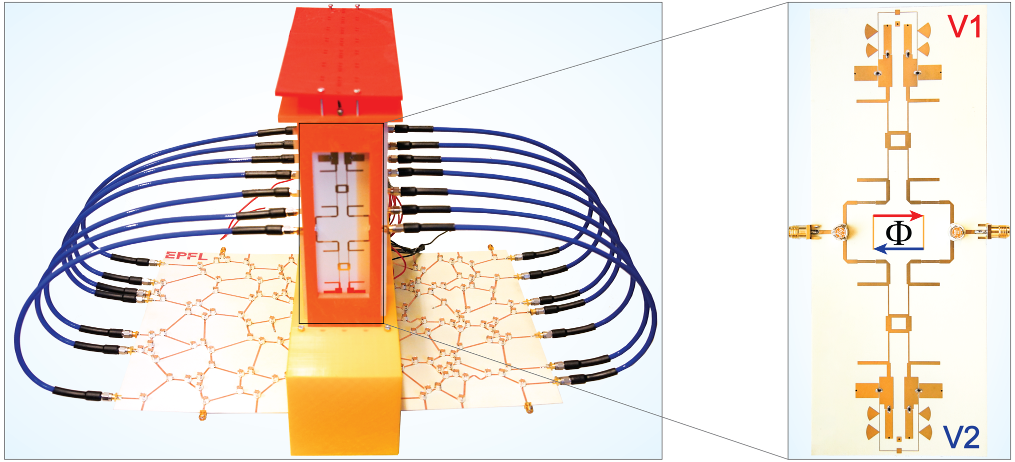
Zhang, Delplace, Fleury, Nature 598, 293 (2022)



Topological invariant measurements

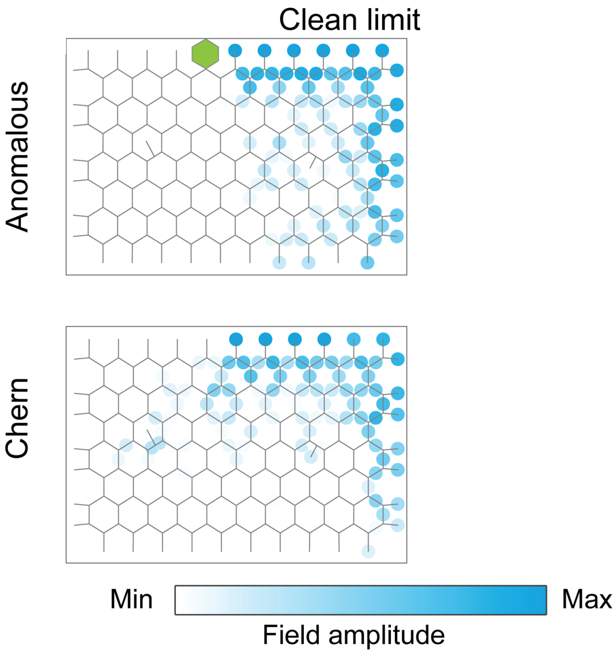


Φ	Non-reciprocal phase
	Twisted boundary condition
🟩	System probe
↑	Incident state |a>
↓	Scattering state |b> = R|a>



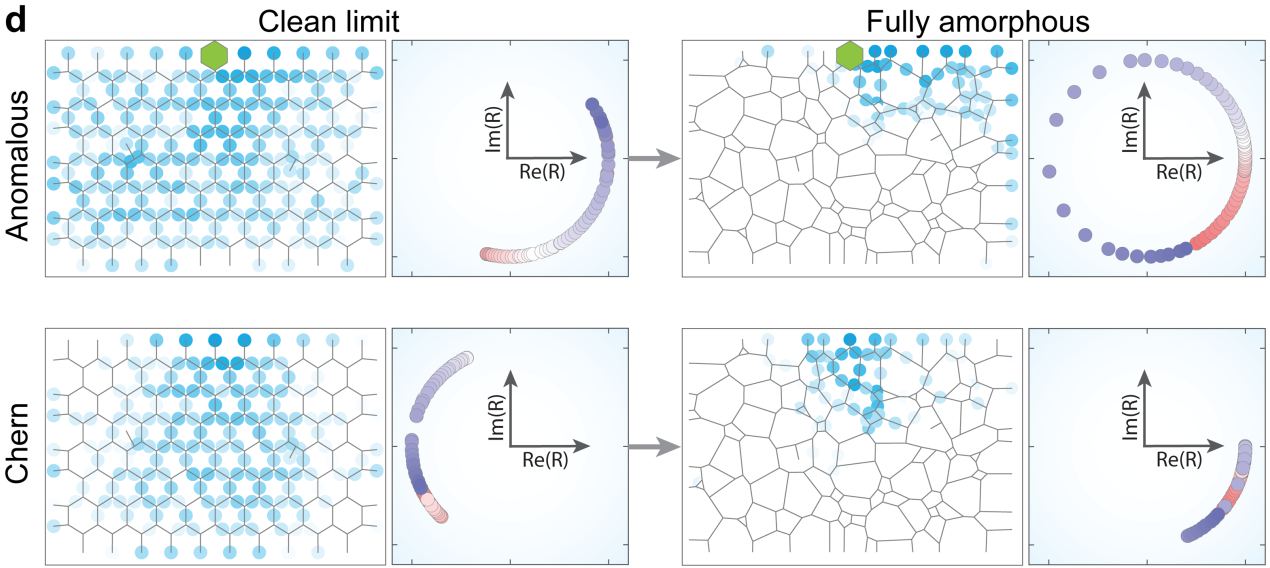
Zhang, Delplace, Fleury, Science Advances 9,eadg3186 (2023),

Topological invariant measurements

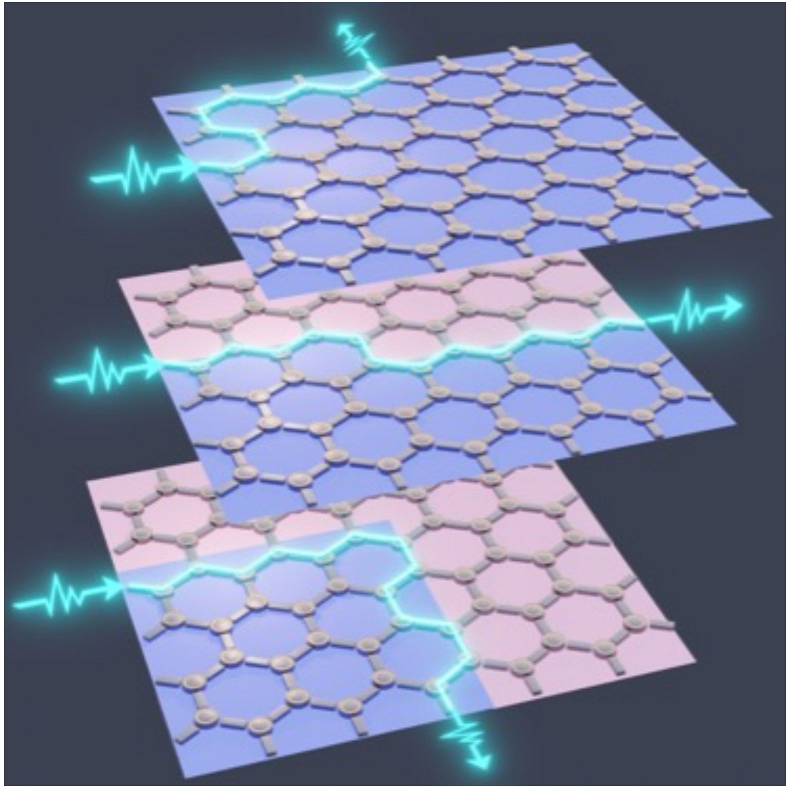


Zhang, Delplace, Fleury, Science Advances 9,eadg3186 (2023),

Topological invariant measurements



Zhang, Delplace, Fleury, Science Advances 9,eadg3186 (2023),



Introduction to topology

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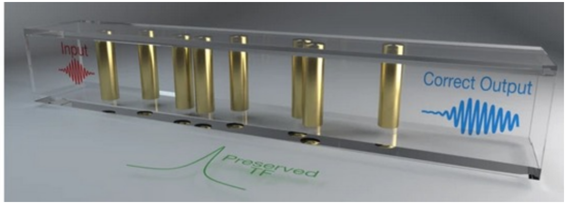
Some take-home messages

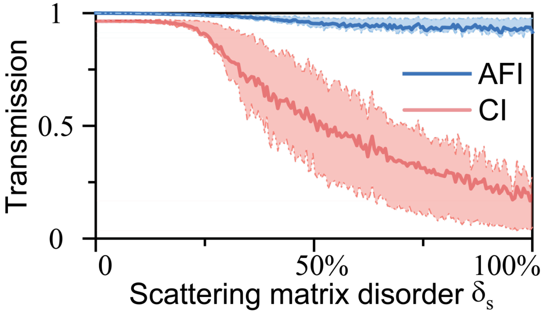
Topological waves

Topology can garantee nice wave properties:
-mode presence
-scattering resonance
-unidirectional transport

Robust wave scattering

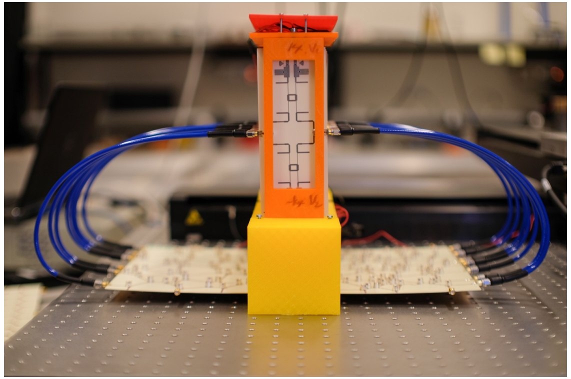
Topological insulators can be used to create scattering signatures/transport channels that can be very robust. Applications ?





Measuring scattering invariants

Practical proof of topology
From a theory-driven field to an experimentally-driven one ?



Refs: Zhang, Delplace, Fleury, Science Advances 9,eadg3186 (2023), Nature 598, 293 (2021)

EPFL

To all my team, current and past students and postdocs, and collaborators!

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Thank you

École
polytechnique
fédérale
de Lausanne